

Consensus of population systems with community structures

Jing Wang,^{1,*} Bin Wu,¹ Long Wang,^{1,†} and Feng Fu^{2,1,‡}

¹Center for Systems and Control, State Key Laboratory for Turbulence and Complex Systems, College of Engineering, Peking University, Beijing 100871, China

²Program for Evolutionary Dynamics, Harvard University, One Brattle Square, Cambridge, Massachusetts 02138, USA
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Multicommunity population systems may reach a consensus state where the fractions of each species in different communities agree on a common value. In this paper, by analyzing the evolutionary dynamics based on an extended replicator equation incorporating community effects, the consensus problem of population systems with n communities is studied. In particular, the simple case of two communities is investigated in detail. In general, for n communities, a sufficient and necessary condition for population systems to reach a consensus of coexistent state is provided. Regarding the population dynamics for the four different types of games, whether the population systems can achieve consensus is determined. The dynamics of community-structured populations shows richer features than nonstructured populations, and some nontrivial phenomena arising from different community-structured population systems are illustrated with concrete numerical examples.

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I. INTRODUCTION

Evolutionary game theory, pioneered by John Maynard Smith [1], offers a powerful and effective framework for studying evolutionary dynamics, and has been widely used in various areas, including biology [2], economics [3], and social sciences [4]. One fundamental mathematical approach to modeling evolutionary game dynamics is based on deterministic differential equations for infinite, well-mixed populations, where every individual interacts with every other individual equally likely. The most classical equations are the Lotka-Volterra [5,6] and replicator equations [7], which are actually mutually equivalent [8]. In addition, classical stochastic processes used in genetics, for example, the Moran process, have provided the basis for an entirely new analysis of evolutionary dynamics in finite populations [9–13]. Both methods consider the evolutionary dynamics of well-mixed populations, allowing one to gain analytical insight.

Recently, there has been a growing interest in studying the evolutionary dynamics of structured populations where individuals are arranged on a network (or graph) [14]. Indeed, it is plausible to take account of limited local, rather than global, interactions and competitions using various network models. More recently, a number of studies have revealed that the outcome of evolutionary games depends crucially on network topologies [15–17]. A variety of network models have been investigated, such as regular graphs (lattices) [18–21], random graphs [16], small-world networks [22], and scale-free networks [23–26]. Nevertheless, it is worth noting that the community structures, which are common in the real world, may also have significant influences on the evolutionary dynamics. For instance, Chen *et al.* introduced a community-structured prisoner's dilemma model to study

the community effects on the evolution of cooperation by simulations and found that changing the ratio between the intercommunity and innercommunity links can affect the dynamical behavior of the population [27]. Here, the so-called community is a group of agents with a high probability of interaction within them, whereas a low probability of interaction between groups [28]. In particular, community property is a signature of the hierarchical nature of complex systems as diverse as real social networks [28–33], information communication [34–36], as well as biology [37–39].

Yet, investigating the effects of community structures on evolutionary dynamics has received little attention. In addition, to our knowledge, there are few analytical results on the community-structured populations. Motivated by these, in this paper, we introduce a minimum yet simple community-structured population model, which mimics the essence of ubiquitous community structures in the real social situations. To this end, we consider a community-structured population consisting of two strategies A and B . Inside each community, individuals contact each other with equal probability, while individuals belonging to different communities interact each other with a lower probability. Interestingly, it is found that such community-structured populations may reach a consensus state, where the fractions of each species in different communities agree on a common value ultimately. Let us consider the simple case of two communities for an illustrative example. If the stationary fractions of A players in both communities are equal, meanwhile, the same case is with B players, we say a consensus state of the population systems is achieved. It is noteworthy that such consensus problems arising in different contexts are also attracting growing attention in the field of control theory [34,35,40–43]. Herein, our main contribution is to investigate the consensus problem in a biological context by considering community-structured populations based on evolutionary game theory. To do this, we develop an extended replicator equation incorporating the effects of community structures, and then analyze the evolutionary dynamics in terms of equilibrium stability of the fixed points. To be concrete, for populations with two com-

*wj02@pku.edu.cn

†longwang@pku.edu.cn

‡fengfu@fas.harvard.edu

munities, we classify four scenarios according to different payoff matrices (i.e., different types of games), and discuss the corresponding conditions for the system to reach consensus. In general, for n communities, we present a sufficient and necessary condition for the system to achieve the consensus state in which A and B players can coexist in each community. Furthermore, through some numerical examples, we illustrate that, in comparison with nonstructured populations, the effects of community structures lead to more diverse dynamic features. For example, the system can split into distinct groups of communities with the same state, depending on the configurations of the community structures.

The remainder of this paper is organized as follows. The replicator equation with n communities is introduced in Sec. II. A complete classification of the dynamics and consensus of the population system with two communities are presented in Sec. III. Regarding the four different types of games, consensus of the population system with n communities is studied in Sec. IV. We provide some numeric examples in Sec. V and finally draw conclusions in Sec. VI.

II. REPLICATOR EQUATION OF THE POPULATION WITH n COMMUNITIES

Let us consider a population consisting of n communities. In each community, individuals have tight connections while between communities, there are only looser connections. Initially, every individual takes part in one of the n communities randomly, resulting in all communities of approximately the same size. Therefore, each individual in the population has two tags. One is the strategy adopted by the individual, and the other is the community which the individual belongs to. In each step of the evolutionary process, everyone can update its strategy but cannot travel to other communities. Hence, the size of each community remains constant and the members in each community cannot be altered.

In our model, we consider a symmetrical 2×2 game with two pure strategies A and B . An A player receives payoff a or b playing against another A or B player; similarly, a B player receives payoff c or d playing against A or B player. We denote by x_i and y_i ($i=1, 2, \dots, n$) the fraction of individuals adopting strategy A and B in community i , respectively. Obviously, we have $x_i + y_i = 1$. The state of the population with n communities represented by a vector (x_1, x_2, \dots, x_n) corresponds to a point in the simplex $S_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$. Individuals play games in terms of pairwise interactions specified by the matrix W with element w_{ij} , $1 \leq i, j \leq n$, which characterizes the interaction probability with which the individual from community i interacts with the one from j . The interaction probability matrix (IPM) W has three properties as listed below.

(i) Elements of the matrix must be real numbers in the closed interval $[0, 1]$ and the sum of elements in each row equals 1, i.e., $0 \leq w_{ij} \leq 1$ and $\sum_{j=1}^n w_{ij} = 1$.

(ii) We suppose that the interaction probabilities between two players are symmetric, i.e., $w_{ij} = w_{ji}$.

(iii) The community structures imply that the interaction probability within the same community is larger than that between communities, i.e., $w_{ii} > w_{ij}$, for all $j \neq i$.

Accordingly, the fitness of A and B players in community i are given by

$$f_{Ai} = a(W\vec{x}^T)_i + b(W\vec{y}^T)_i,$$

$$f_{Bi} = c(W\vec{x}^T)_i + d(W\vec{y}^T)_i.$$

Here, $\vec{x} = (x_1, x_2, \dots, x_n)$ and $\vec{y} = (y_1, y_2, \dots, y_n)$ are n -dimensional row vectors. In each community, we use the standard replicator equation which takes the form of

$$\dot{x}_i = x_i(f_{Ai} - \bar{f}_i),$$

$$\dot{y}_i = y_i(f_{Bi} - \bar{f}_i), \quad (1)$$

where $\bar{f}_i = x_i f_{Ai} + y_i f_{Bi}$ is the average fitness of individuals in the community i .

Substituting $y_i = 1 - x_i$ into the first equation of Eq. (1), we obtain a simplified form

$$\dot{x}_i = x_i(1 - x_i)[\alpha(W\vec{x}^T)_i + \beta], \quad (2)$$

where $\alpha = a - b - c + d$, $\beta = b - d$. Note that the population dynamics expressed by Eq. (2) is no longer a standard replicator equation. Here, we call Eq. (2) as an extended replicator equation.

III. CONSENSUS OF POPULATION SYSTEMS WITH TWO COMMUNITIES

Let us turn to the simple case of two communities. In this situation, the IPM W is dependent of only one element. For simplicity, we use ρ to denote the element w_{11} , then the IPM W reads

$$\begin{bmatrix} \rho & 1 - \rho \\ 1 - \rho & \rho \end{bmatrix}.$$

From Eq. (2), we obtain nine possible equilibria $(0, 0)$, $(0, 1)$, $(0, -\beta/\rho\alpha)$, $(1, 0)$, $(1, 1)$, $\{1, -[(1-\rho)\alpha + \beta]/\rho\alpha\}$, $(-\beta/\rho\alpha, 0)$, $\{-[(1-\rho)\alpha + \beta]/\rho\alpha, 1\}$ and $(-\beta/\alpha, -\beta/\alpha)$. Note that only $(-\beta/\alpha, -\beta/\alpha)$ is an interior equilibrium and all the other points are boundary equilibria. If the interior equilibrium does not exist, every orbit of the replicator equation converges to the boundary of the simplex S_2 (for populations with two communities, the readers can refer to Table I in Appendix A for details about the stability of each equilibrium). According to the comparison between a and c and that between b and d , we distinguish four types of games as listed below.

A. A dominates B

If $a > c$, $b > d$, then strategy A dominates B in nonstructured populations. The corresponding representative game is the prisoner's dilemma (PD) game in which strategies A and B denote defection and cooperation, respectively. The Nash equilibrium condition ensures that no player has a unilateral incentive to deviate and play another strategy, as there is no way he would be better off, given the others' choices. For populations without community structures, it is best to play

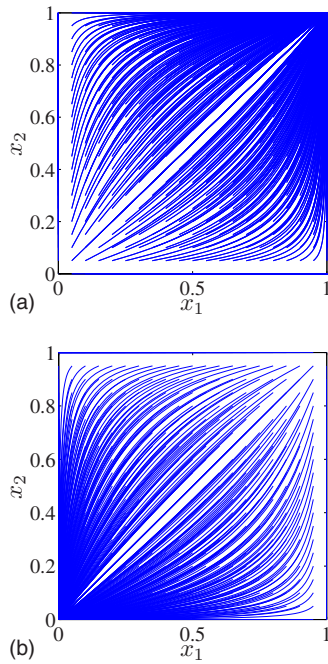


FIG. 1. (Color online) Phase diagrams for the case of dominant A (B). (a) For $a > c$ and $b > d$ ($a=5, b=1, c=3, d=0, \rho=0.6$), the fixed points $(0,0), (0,1), (1,0)$, and $(1,1)$ are unstable, saddle, saddle, and stable nodes, respectively. This means that A dominates B. The population system can reach consensus irrespective of the initial state and the interaction probability matrix. (b) For $a < c$ and $b < d$ ($a=3, b=0, c=5, d=1, \rho=0.6$), the fixed points $(0,0), (0,1), (1,0)$, and $(1,1)$ are stable, saddle, saddle, and unstable nodes, respectively. Thus, strategy B dominates A. The system can arrive at consensus of all B.

the strategy of defection (A), as the “all defectors” set of strategies is the unique Nash equilibrium in this game. For populations with community structures, however, the evolutionary dynamics is not changed by introducing the community structures. The reason is that in each community, the fitness of defectors, which is the best response for every individual, is always larger than that of cooperators. Therefore, in the whole populations, defection (A) is still the best choice. In this case, the system ends up with the state of all A, i.e., consensus can be reached, irrespective of the initial state and the IPM W (for details, please see Appendix B). The phase diagram of the evolutionary dynamics is depicted in Fig. 1(a).

B. B dominates A

If $a < c$ and $b < d$, strategy B dominates A in nonstructured populations. This case is analogous with the above one. The phase diagram can be found in Fig. 1(b).

C. A and B coexist

For $a < c$ and $b > d$, the representative model is the hawk-dove game (also called the snowdrift or chicken game). For nonstructured populations, the strategy profiles (A,B) and (B,A) are Nash equilibrium in this game. Thus, it is best to play the strategy opposite to one’s opponent, resulting in

coexistence of A and B at a mixed stable state where the fraction of A players is $x_A^* = -\beta/\alpha$. For populations with two communities, although the convergent orbit is more complex, the final state of the system also converges to the interior equilibrium $(-\beta/\alpha, -\beta/\alpha)$. This is because, in analogy to the situation of nonstructured populations, for any community i which arrives at stable equilibrium state, it requires $f_{Ai} = f_{Bi}$, i.e., $\sum_{j=1}^n w_{ij}x_j = -\beta/\alpha$. This condition holds uniquely when $x_1^* = -\beta/\alpha$ and $x_2^* = -\beta/\alpha$ in the case of two communities. As a result, A and B can coexist at the same mixed equilibrium in each community, resulting in a consensus state of the population system. In addition, we should point out that although the evolutionary outcome cannot be altered, the convergent orbit is affected by the IPM W (ρ) and the payoff matrix (see Fig. 2). The detailed analysis of all fixed points is provided in Appendix B.

D. A and B are bistable

If $a > c$ and $b < d$, the payoff matrix denotes the coordination game in which each strategy is the best reply to itself. Hence, both strategy profiles (A,A) and (B,B) are Nash equilibria, making strategies A and B bistable in the nonstructured populations. Moreover, which equilibrium the population evolves to depends on the interior fixed point $x_A^* = -\beta/\alpha$ and the initial fraction of A. If the initial fraction of A is larger than $-\beta/\alpha$, the population will evolve to all A, otherwise, to all B.

For populations with two communities, the evolutionary dynamics is altered largely by the community structures. The community with the initial fraction of A less than $-\beta/\alpha$ may still end up with all A, in sharp contrast to the case of nonstructured populations. The reasoning is that, although the initial fraction of A in a certain community i is less than $-\beta/\alpha$, the sum $\sum_{j=1}^n w_{ij}x_j$ may exceed $-\beta/\alpha$ under the influence of other communities. This effect leads to the increase of A players in this community, further paving the way for community i to converge to the state of all A. Accordingly, whether a community can converge to all A (or all B) depends on the value of $x_A^* = -\beta/\alpha$ and the combined effect of interaction probabilities and the state of the population. Specifically, if the sum $\sum_{j=1}^n w_{ij}x_j$ is larger than $-\beta/\alpha$, the fraction of A players in community i is increased, otherwise, is decreased. Thus, community structure plays a nontrivial role on the population dynamics. It is worth noting that, in the case of $\alpha + 2\beta < 0$ and $\rho > (a-c)/\alpha$, in addition to the stable fixed points $(0,0)$ and $(1,1)$, the boundary fixed points $(0,1)$ and $(1,0)$ are also stable, namely, the situation of one community consisting of all A and the other of all B is conceivable [see Fig. 3(c)], which deviates from the definition of consensus. Therefore, for the coordination games, population systems can reach consensus only when the initial state of the population lies in the attraction basins of $(0,0)$ and $(1,1)$ (region D_1 and D_2 in Fig. 3). The phase diagrams corresponding to different arrangements of parameters are plotted in Fig. 3.

IV. CONSENSUS OF POPULATION SYSTEMS WITH n COMMUNITIES

Consider the population with n communities. According to the payoff matrix, we also distinguish four different types of games.

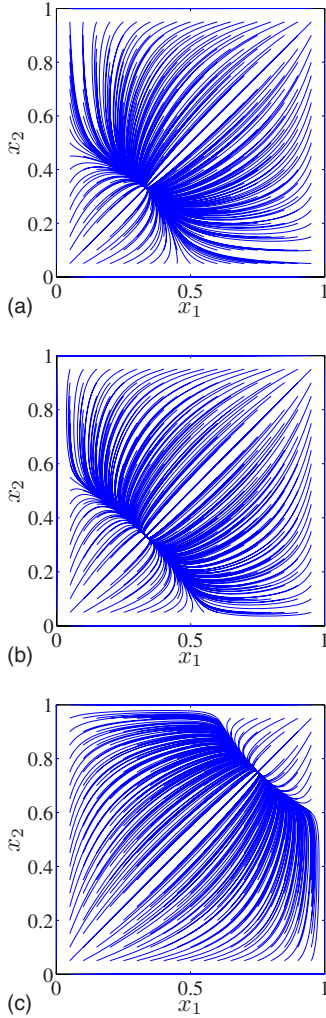


FIG. 2. (Color online) Phase diagrams of the situation where A and B coexist. (a) If $\alpha+2\beta < 0$ and $\rho > (a-c)/\alpha$ ($a=3, b=1, c=5, d=0, \rho=0.7$), all orbits converge to the interior equilibrium $(1/3, 1/3)$. Points $(0,0), (0,1), (1,0)$, and $(1,1)$ are all unstable nodes, and the rest of the four equilibrium points are all saddle nodes. (b) If $\alpha+2\beta < 0$ and $-\beta/\alpha < \rho < (\alpha+\beta)/\alpha$ ($a=3, b=1, c=5, d=0, \rho=0.6$), all orbits still converge to the interior equilibrium $(1/3, 1/3)$. Only two boundary equilibrium points $(0,0)$ and $(1,1)$ are unstable nodes. All the other equilibria are saddle nodes except points $(1, -1/9)$ and $(-1/9, 1)$ which are not found in the plane. (c) If $\alpha+2\beta > 0$ and $(\alpha+\beta)/\alpha < \rho < -\beta/\alpha$ ($a=1, b=3, c=2, d=0, \rho=0.6$), the population still arrives at a coexistence state in the interior equilibrium $(0.75, 0.75)$, resembling the case (b). In all the above cases, the population systems can reach a consensus state of coexistence.

A. Consensus under the condition of $a > c$ and $b > d$ (or $a < c$ and $b < d$)

For a game with $a > c$ and $b > d$ (or $a < c$ and $b < d$), a representative example is the PD game. In this case, we have shown that every orbit of Eq. (2) in two dimensions converges to the boundary of S_2 . For population systems with n communities ($n > 2$), the dynamics is similar to that of $n = 2$. The fixed point $(-\alpha/\beta, \dots, -\alpha/\beta)$ is not in S_n , so that every orbit also goes to the boundary of S_n . Moreover, by the

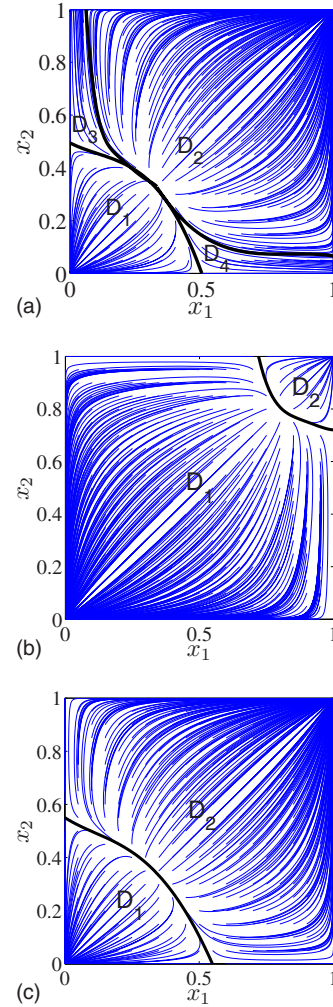


FIG. 3. (Color online) Phase diagrams of the situation where A and B are bistable. The interior equilibrium is always unstable in below situations. (a) For $\alpha+2\beta < 0$ and $\rho > -\beta/\alpha$ ($a=5, b=0, c=3, d=1, \rho=0.7$), all trajectories converge to one of the four stable fixed points $(0,0), (0,1), (1,0)$ as well as $(1,1)$, and the other four equilibrium points are all saddle nodes. In this case, population systems may achieve consensus. (b) For $\alpha+2\beta < 0$ and $(\alpha+\beta)/\alpha < \rho < -\beta/\alpha$ ($a=5, b=0, c=4, d=4, \rho=0.7$), all orbits converge to either of the two equilibria $(0,0)$ and $(1,1)$. Fixed points $(0, -\beta/\rho\alpha)$ and $(-\beta/\rho\alpha, 0)$ cannot be observed in the phase plane and the other four equilibria are all saddle nodes. Population systems can reach consensus irrespective of the initial state. (c) For $\alpha+2\beta > 0$ and $-\beta/\alpha < \rho < (\alpha+\beta)/\alpha$ ($a=5, b=0, c=3, d=1, \rho=0.6$), population systems still reaches consensus state independent of the initial state. The other fixed points are all saddle nodes except the equilibria $\{1, -[(1-\rho)\alpha+\beta]/\rho\alpha\}$ and $\{-[(1-\rho)\alpha+\beta]/\rho\alpha, 1\}$ which are not in the phase plane. In the above cases, region D_1, D_2, D_3 , and D_4 represent the attraction basins of $(0,0), (1,1), (0,1)$, and $(1,0)$, respectively.

signs of the eigenvalues of the Jacobian matrix at the fixed points, it is easy to verify that in the case of $a > c$ and $b > d$, the equilibrium $(0, \dots, 0)$ is unstable and $(1, \dots, 1)$ is stable; in the case of $a < c$ and $b < d$, the former is stable and the latter is unstable. In addition, the other boundary fixed points are all saddles. Hence, in the above cases, only one type of strategy can survive the game, which makes the

population consist of the same players. Specifically, for the PD game, defection is still the best choice for each individual in all communities, enabling defection to spread all over the whole population. Thus, population systems can evolve to the state of all A (B) under the condition of $a > c$ and $b > d$ (or $a < c$ and $b < d$), resulting in a consensus.

B. Consensus under the condition of $a < c$ and $b > d$ (or $a > c$ and $b < d$)

If $a < c$ and $b > d$ (the representative example is the hawk-dove game) or $a > c$ and $b < d$ (coordination games), the population ends up with coexistent or homogeneous state, respectively, for the case of $n=2$. In the case of $n > 2$, it is easy to obtain that the equilibrium $(0, \dots, 0)$ and $(1, \dots, 1)$ are unstable for $a < c$ and $b > d$, while both stable for $a > c$ and $b < d$. Population systems are not able to reach consensus of all A or all B for the former type of games. In contrast, they can obtain such consensus for the latter games. Note that population systems cannot reach consensus at other boundary fixed points except $(0, \dots, 0)$ and $(1, \dots, 1)$, we do not concentrate on the other boundary fixed points. As consensus may be achieved at interior fixed point, we focus on analyzing the stability of the interior equilibrium.

Theorem 1. The stability of the interior equilibrium of Eq. (2) is equivalent to that of the equation

$$\dot{z}_i = z_i(\alpha(Wz^T)_i + \beta), \quad i = 1, \dots, n, \quad (3)$$

where \vec{z} is the corresponding n -dimensional row vector. The proof of theorem 1 is provided in Appendix C.

According to theorem 1, removing the term $(1-x_i)$ only influences the convergent rate. In order to investigate the stability of the interior equilibrium of Eq. (2), we analyze the corresponding simplified dynamics represented by Eq. (3), which is a standard Lotka-Volterra equation for n species.

Lemma 1. (See theorem 7.5.1 in Ref. [8]) There exists a differentiable map from $S_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$ onto $\hat{S}_{n+1} = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^{n+1} x_i = 1\}$ mapping the orbits of the Lotka-Volterra equation (3) onto the orbits of the replicator equation

$$\dot{\xi}_i = \xi_i[(Q\vec{\xi}^T)_i - \vec{\xi}Q\vec{\xi}^T], \quad i = 1, \dots, n+1, \quad (4)$$

where $\vec{\xi}$ is an $(n+1)$ -dimensional row vector and the matrix Q is given below:

$$\begin{bmatrix} \alpha w_{11} + \beta & \alpha w_{12} + \beta & \cdots & \alpha w_{1n} + \beta & \beta \\ \alpha w_{21} + \beta & \alpha w_{22} + \beta & \cdots & \alpha w_{2n} + \beta & \beta \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha w_{n1} + \beta & \alpha w_{n2} + \beta & \cdots & \alpha w_{nn} + \beta & \beta \\ \beta & \beta & \cdots & \beta & 0 \end{bmatrix}.$$

Essentially, the above matrix Q can be viewed as an $(n+1) \times (n+1)$ payoff matrix for the game with $n+1$ types of strategies. The $Q_{i,j}$ denotes the payoff of an individual playing strategy i against the one with strategy j . Accordingly, we obtain a result in the following theorem.

Theorem 2. The stability of the interior equilibrium of a population consisting of two species with n communities is

equivalent to that of a population composed of $n+1$ species but without community structures.

It should be stressed that for an invertible IPM W , the interior equilibrium point of Eq. (4) is

$$\vec{\xi}^* = \left(\frac{\beta}{n\beta - \alpha}, \frac{\beta}{n\beta - \alpha}, \dots, \frac{\beta}{n\beta - \alpha}, \frac{\alpha}{\alpha - n\beta} \right).$$

Note that in the cases of $a > c$ and $b > d$ and $a < c$ and $b < d$, the fixed point $(-\alpha/\beta, \dots, -\alpha/\beta)$ does not exist in S_n . Consequently, in these two cases, theorem 1 and lemma 1 are both invalid. However, in the cases of $a < c$ and $b > d$ and $a > c$ and $b < d$, the interior equilibrium $\vec{\xi}^*$ of Eq. (4) are in \hat{S}_{n+1} , in accordance with the valid interior equilibrium \vec{x}^* of Eq. (2). Hence, in the cases of $a < c$ and $b > d$ and $a > c$ and $b < d$, the stability of \vec{x}^* can be derived from that of $\vec{\xi}^*$. Thus, we obtain a sufficient and necessary condition for the stability of the interior fixed point.

Theorem 3. The interior equilibrium $\vec{\xi}^* = (\frac{\beta}{n\beta - \alpha}, \dots, \frac{\beta}{n\beta - \alpha}, \frac{\alpha}{\alpha - n\beta})$ is asymptotically stable if and only if for any $\vec{\xi} \neq \vec{\xi}^*$ in some neighborhood of $\vec{\xi}^*$ in \hat{S}_{n+1} , the inequality

$$\vec{\xi} \begin{bmatrix} \alpha W & 0 \\ 0 & -\beta \end{bmatrix} \vec{\xi}^T + \frac{\alpha\beta}{\alpha - n\beta} < 0 \quad (5)$$

always holds.

Proof. Since the payoff matrix Q is symmetric, the game is a partnership game, which is also known as “game with identical interests” or “potential game” [44]. By theorem 7.8.1 in Ref. [8], we know that the interior equilibrium $\vec{\xi}^*$ is an asymptotically stable state if and only if it is also an evolutionarily stable state which needs the inequality $\vec{\xi}Q\vec{\xi}^T < \vec{\xi}^*Q\vec{\xi}^T$ to be satisfied for any $\vec{\xi} \neq \vec{\xi}^*$ in some neighborhood of $\vec{\xi}^*$ in \hat{S}_{n+1} .

Furthermore, we obtain that

$$\vec{\xi}^*Q\vec{\xi}^T = -\frac{n\beta^2}{\alpha - n\beta}$$

independent of the vector $\vec{\xi}$. In addition, we also have

$$\vec{\xi}Q\vec{\xi}^T = \beta + \vec{\xi} \begin{bmatrix} \alpha W & 0 \\ 0 & -\beta \end{bmatrix} \vec{\xi}^T.$$

The inequality $\vec{\xi}Q\vec{\xi}^T < \vec{\xi}^*Q\vec{\xi}^T$ yields that

$$\vec{\xi} \begin{bmatrix} \alpha W & 0 \\ 0 & -\beta \end{bmatrix} \vec{\xi}^T < \frac{\alpha\beta}{n\beta - \alpha}.$$

■

Based on inequality (5), we can analyze the stability of the interior equilibrium of Eq. (2) in n dimensions. If inequality (5) is satisfied, every orbit of Eq. (2) converges to the interior equilibrium \vec{x}^* which is a sink. This results in coexistence of A and B in each community and a consensus state of population systems. If inequality (5) does not hold, the interior equilibrium \vec{x}^* is neither evolutionarily stable state nor asymptotically stable state. In this case, the interior fixed point is a source and all orbits move away from it.

Population systems cannot attain the consensus of coexistent state. Thus, inequality (5) is also a sufficient and necessary condition to judge whether or not population systems can reach the consensus in which *A* and *B* can coexist. From theorem 3, we get a further result as follows.

Corollary 1. In the case of $a > c$ and $b < d$, the interior equilibrium $\vec{\xi}^* = (\frac{\beta}{n\beta-\alpha}, \frac{\beta}{n\beta-\alpha}, \dots, \frac{\beta}{n\beta-\alpha}, \frac{\alpha}{\alpha-n\beta})$ is a source.

Proof. It is easy to find a point $\vec{\xi} = (1, 0, \dots, 0)$, $\vec{\xi} \neq \vec{\xi}^*$ in some neighborhood of $\vec{\xi}^*$ in \hat{S}_{n+1} . We have

$$\vec{\xi} \begin{bmatrix} \alpha W & 0 \\ 0 & -\beta \end{bmatrix} \vec{\xi}^T = \alpha w_{11}.$$

In the case of $a > c$ and $b < d$, there are $\alpha > 0$, $\beta < 0$ and $\alpha > -\beta$. For $n > 1$, we obtain $\alpha w_{11} > \alpha\beta/(n\beta-\alpha)$, derived from $\beta/(n\beta-\alpha) < 1/n$. As inequality (5) is not satisfied, the interior equilibrium is neither evolutionarily stable state nor asymptotically stable state. It is a source. ■

Therefore, in the case of $a > c$ and $b < d$, all orbits of Eq. (4) drive away from the interior fixed point and converge to the boundary of \hat{S}_{n+1} , i.e., the systems cannot reach consensus where both strategies coexist. However, as the boundary fixed points $(0, \dots, 0)$ as well as $(1, \dots, 1)$ are stable, the whole population may end up with the state of all *A* or all *B*. For the case of $a < c$ and $b > d$, we give another condition which is different than theorem 3 to judge the stability of the interior equilibrium.

Theorem 4. In the case of $a < c$ and $b > d$, the interior equilibrium $\vec{\xi}^* = (\frac{\beta}{n\beta-\alpha}, \frac{\beta}{n\beta-\alpha}, \dots, \frac{\beta}{n\beta-\alpha}, \frac{\alpha}{\alpha-n\beta})$ is a stable node for a positive definite IPM *W*.

Proof. Let

$$\bar{Q} = \begin{bmatrix} \alpha W & 0 \\ 0 & -\beta \end{bmatrix}.$$

The stability of Eq. (4) is the same with that of the equation

$$\dot{\xi}_i = \xi_i [(\bar{Q}\vec{\xi}^T)_i - \vec{\xi}\bar{Q}\vec{\xi}^T], \quad i = 1, \dots, n + 1.$$

In the case of $a < c$ and $b > d$, we have the parameters $\alpha < 0$ and $\beta > 0$. As the IPM *W* is a positive definite matrix, the matrix \bar{Q} is a negative definite matrix, giving rise to the Lyapunov function $V(\vec{\xi}) = -\vec{\xi}\bar{Q}\vec{\xi}^T > 0$. Since \bar{Q} is symmetric, we obtain

$$\begin{aligned} \dot{V}(\vec{\xi}) &= -2\vec{\xi}\bar{Q}\vec{\xi}^T = -2\sum_{i=1}^{n+1} \xi_i (\bar{Q}\vec{\xi}^T)_i \\ &= -2\sum_{i=1}^{n+1} \xi_i (\bar{Q}\vec{\xi}^T)_i^2 + 2\left(\sum_{i=1}^{n+1} \xi_i (\bar{Q}\vec{\xi}^T)_i\right)^2. \end{aligned}$$

As

$$\sum_{i=1}^{n+1} \xi_i = 1,$$

we have

$$\dot{V}(\vec{\xi}) = -2\sum_{i=1}^{n+1} \xi_i [(\bar{Q}\vec{\xi}^T)_i - \vec{\xi}\bar{Q}\vec{\xi}^T]^2,$$

leading to

$$\dot{V}(\vec{\xi}) \leq 0.$$

So the interior rest point $\vec{\xi}^*$ is a stable node. ■

That is, for a game with the payoff matrix satisfying $a < c$ and $b > d$, if the IPM *W* is a positive definite matrix, the interior equilibrium of the population with *n* communities is always a sink, which attracts all orbits in S_n . Each community stays in the coexistent state of *A* and *B* eventually and the final fractions of any one species in each community agree on a common value. The consensus of coexistence can be achieved.

V. INFLUENCES BY INTRODUCING THE COMMUNITY STRUCTURE

Based on the above discussion, for the games with $a > c$ and $b > d$ as well as $a < c$ and $b < d$, the evolutionary dynamics of populations with community structures remains qualitatively the same as that of nonstructured populations. Population systems can achieve the consensus of homogeneous state which is full of the same players. However, for the games with $a < c$ and $b > d$ and those with $a > c$ and $b < d$, the impacts of community structures on evolutionary dynamics are complicated, bringing in diverse dynamic behaviors of population systems. We shall illustrate these two cases for $n > 2$ by the following concrete examples.

A. Hawk-dove game

Consider a population with hawk and dove strategies [1]. Hawk is a strategy which fights until one of the players sustains an injury or the opponent retreats. If a hawk meets a hawk, there is a combat resulting in half of the gain and half of the loss for each hawk, i.e., $(G-C)/2$. When a dove meets a hawk, it immediately concedes the field to the hawk. Hence, the hawk and the dove receive payoff *G* and 0, respectively. When a dove meets another dove, both of them win half of the gains without cost, i.e., $G/2$. The corresponding payoff matrix of the hawk-dove game is given by

	Hawk	Dove
Hawk	$(G-C)/2$	<i>G</i>
Dove	0	$G/2$

Note that the elements $(G-C)/2$, *G*, 0 and $G/2$ correspond to *a*, *b*, *c*, and *d* in the above discussion. In the hawk-dove games, we always assume that the cost *C* is larger than the prize *G*.

For nonstructured populations, the system ends up with a coexistent state where the fraction of hawk is G/C ultimately regardless of the initial state. For community-structured populations, the effects of community structures on the dynamics cannot be ignored. Suppose that there are six communities in the population and the IPM *W* is given as follows:

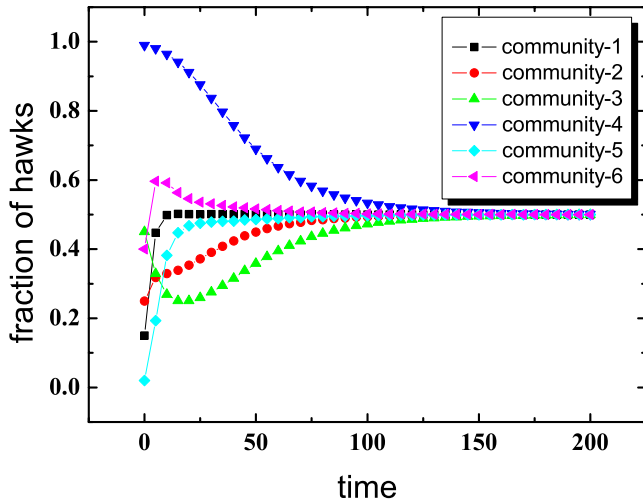


FIG. 4. (Color online) Population systems with the Hawk-Dove game can reach consensus. Let the prize $G=2$ and the cost $C=4$. We get the parameter $\alpha=-C/2=-2$ and $\beta=G/2=1$. The initial state of the population is $(0.15, 0.25, 0.45, 0.99, 0.02, 0.4)$. We can find that each community's state converges to a common value.

$$\begin{bmatrix} 0.8 & 0.05 & 0.05 & 0.05 & 0.05 & 0 \\ 0.05 & 0.3 & 0.2 & 0.2 & 0.1 & 0.15 \\ 0.05 & 0.2 & 0.4 & 0.3 & 0 & 0.05 \\ 0.05 & 0.2 & 0.3 & 0.35 & 0.05 & 0.05 \\ 0.05 & 0.1 & 0 & 0.05 & 0.6 & 0.2 \\ 0 & 0.15 & 0.05 & 0.05 & 0.2 & 0.55 \end{bmatrix},$$

which is a positive definite matrix. According to theorem 4, each community converges to a coexistent state in which both hawk and dove survive, independent of the initial state (see Fig. 4). Furthermore, as the interior equilibrium is $(G/C, G/C, \dots, G/C)$, hawks achieve the same ratio G/C ($1-G/C$ for doves) among all communities. Population systems reach the consensus state where coexistence of hawk and dove is permitted.

However, for a noninvertible IPM W , Eq. (2) no longer has an unique interior equilibrium, but a subset of interior fixed points. Thus, the point $(-\beta/\alpha, \dots, -\beta/\alpha)$ is not longer a stable interior equilibrium. In particular, population systems split into several groups and communities in each group can come to a common state eventually. Interestingly, how many groups the population splits into depends on the rank of the noninvertible matrix W . Consequently, the population arrives at a splitting state (see Fig. 5). This situation reveals that some specific community structures are an alternative mechanism to maintain a diverse society. In Fig. 5, we adopt the IPM given by

$$\begin{bmatrix} 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{bmatrix}.$$

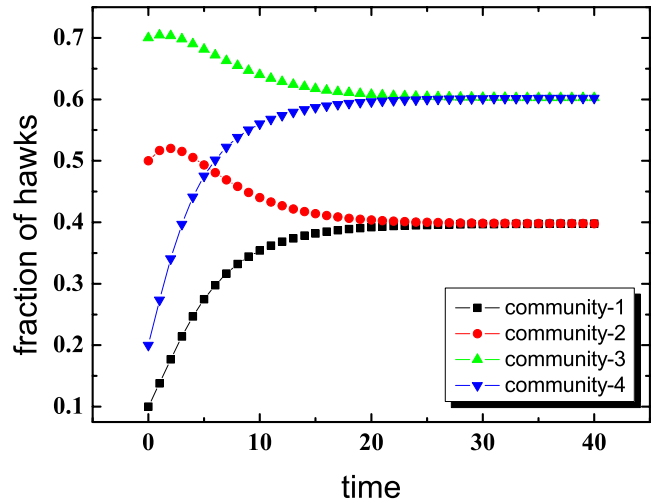


FIG. 5. (Color online) Population systems with the hawk-dove game can reach a splitting state for a noninvertible IPM W . We also use the prize $G=2$ and the cost $C=4$, and set the initial state of the population as $(0.1, 0.5, 0.7, 0.2)$. It is found that the first and the second communities converge to a common state which is less than $-\beta/\alpha$, whereas the third and the fourth one reach a different common state which exceeds $-\beta/\alpha$. The population is divided into two groups and cannot attain consensus.

B. Coordination games

According to the above analysis, the evolutionary dynamics is significantly affected by community structures in the case of $a > c$ and $b < d$. Therefore, different consensus phenomena are yielded by the combined effects of interaction probabilities (elements of W) and the initial state. Suppose there are two choices A and B , in populations. We use the payoff matrix $a=5, b=0, c=3$, and $d=1$. In the coordination games, choosing the same strategy adopted by the opponent is the best option for players. Hence, in nonstructured case, the population ends up with all players choosing the same strategy. Moreover, which state the population converges to, all A or all B , depends on the interior fixed point $x^*=-\beta/\alpha$ (in this example, $x^*=1/3$) and the initial fraction of A players, i.e., if the initial state exceeds $1/3$, the population will be full of A players eventually; otherwise, A players will become extinct.

For community-structured populations, the interior equilibrium $(-\beta/\alpha, \dots, -\beta/\alpha)$ plays an important role in determining the dynamics in the case of an invertible IPM W . In each community, the evolutionary trend is determined not only by the initial fraction of A in this community, but also by those in other communities. For simplicity, we call the trend in the absence of community structures as original evolutionary trend. For example, if in community i , the initial fraction of A is less than $-\beta/\alpha$, the original evolutionary trend is to be the state of all B . Once community structures are presented, combined effects of structures and the initial states of all communities may drive this community to the state of all A , deviating from the original evolutionary trend.

As shown in Fig. 6(a), the initial population state is $(0.32, 0.6, 0.2, 0.5, 0.3, 0.4)$. Although there are three communities where the original trend is to be the state of all B , the whole

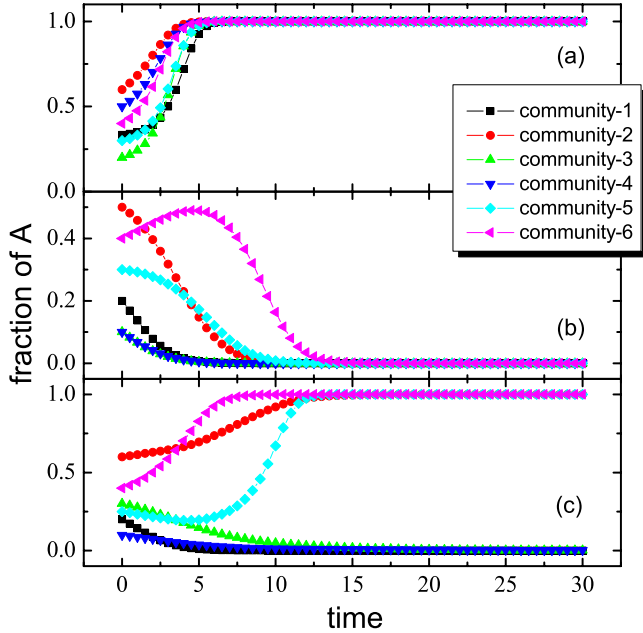


FIG. 6. (Color online) In the case of $a < c$ and $b > d$, dynamics of three different populations. In the three figures, we use the same payoff matrix ($a=5, b=0, c=3, d=1$) (a). Setting the initial state of the population as $(0.32, 0.6, 0.2, 0.5, 0.3, 0.4)$, we get a consensus state of all A. (b) The initial state of the population $(0.2, 0.5, 0.1, 0.1, 0.25, 0.4)$ leads to an opposite consensus state of all B. (c) If the initial state is $(0.2, 0.6, 0.3, 0.1, 0.25, 0.4)$, population systems split into two contrary parts, i.e. one is communities of all A and the other is those of all B. In these three examples, we also use the 6×6 IPM adopted in the Hawk-Dove game mentioned above.

population still evolves to the state of all A ultimately under the influences of community structures. Thus, the population systems can reach consensus state of all A. Conversely, in Fig. 6(b), under the initial condition of $(0.2, 0.5, 0.1, 0.1, 0.25, 0.4)$, though there are communities where the original evolutionary trend is towards the state of all A, all individuals choose B eventually. In this case, population systems can also come to consensus. Interestingly, as shown in Fig. 6(c), some communities end up with all A players while the others with all B. The two choices of A and B can coexist in the whole population, in sharp contrast to the situation without community structures. In this case, the population system cannot arrive at consensus but stays at a splitting state.

How does the combined effect of interaction probabilities and the states of all communities influence the evolutionary dynamics? The explanation can be derived from Eq. (2). When the “combined effect” $\sum_{j=1}^n w_{ij}x_j$ exceeds $x^* = -\beta/\alpha$, the fitness of A players in community i is larger than that of B, increasing the fraction of A in this community; whereas when the combined effect is less than x^* , the fitness of A players in community i is less than that of B, reducing the fraction of A in this community. Take the fifth community in the second case in Fig. 6 for example (see Fig. 7). Initially, as the fraction of A in this community is less than $x^* = -\beta/\alpha$, the original trend is towards the state of all B. The combined effect below $-\beta/\alpha$ leads to the decrease of A players in the fifth community. Then, with the combined effect exceeding

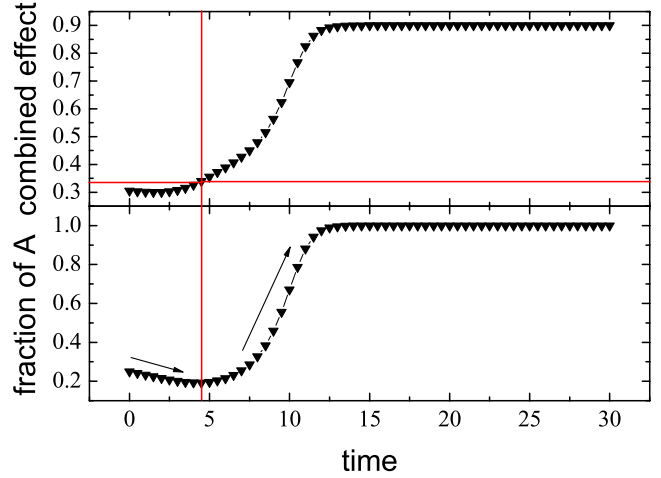


FIG. 7. (Color online) Relationship between the combined effect $\sum_{j=1}^n w_{ij}x_j$ and the fraction of A players in community i . The horizontal red line in the first figure indicates the value $x^* = -\beta/\alpha$ (which is equal to $1/3$ in this example). It shows that the fraction of A players in this community increases when the combined effect exceeds $1/3$, while decreases when the combined effect is less than $1/3$.

$x^* = -\beta/\alpha$, the fraction of A in this community increases until it reaches the state of all A. Thus, the original evolutionary trend of the fifth community is altered by the present community structures and the states of other communities. Overall, these results indicate that community structures of populations have nontrivial effects on the evolutionary dynamics.

VI. CONCLUSIONS

In conclusion, we have presented a theoretical analysis of the population dynamics with community structures based on an extended replicator equation. Regarding the four different types of games quantifying the population dynamics, we investigate the resulting consensus problem in such population systems. We show that, for the case of $a > c$ and $b > d$ ($a < c$ and $b < d$), the system can reach a consensus of homogeneous state. The dynamics in this case remains qualitatively unchanged as compared with the nonstructured populations, although convergent trajectories are naturally affected by the underlying community structures. While, for the case of $a < c$ and $b > d$, the system can achieve a consensus of coexistent state in all communities, if the interaction probability matrix specifying the community structure is positive definite. For noninvertible interaction probability matrices, the system can split into distinct groups of communities with the identical coexistent state. For the case of $a > c$ and $b < d$, in particular, the population in each community tends to be homogeneous eventually, but, the whole population system may turn out to be heterogeneous, in the sense that some communities are full of one type, while the rest consist of the other type. In other words, depending on

TABLE I. Stability of each fixed point

	$a < c$ and $b > d$					
	$a > c,$ $b > d$	$a > c,$ $b < d$	$\alpha + 2\beta < 0,$ $\rho > \frac{\alpha + \beta}{\alpha}$	$\alpha + 2\beta < 0,$ $-\frac{\beta}{\alpha} < \rho < \frac{\alpha + \beta}{\alpha}$	$\alpha + 2\beta > 0,$ $\rho > -\frac{\beta}{\alpha}$	$\alpha + 2\beta > 0,$ $\frac{\alpha + \beta}{\alpha} < \rho < -\frac{\beta}{\alpha}$
(0,0)	unstable	stable	unstable	unstable	unstable	unstable
(0,1)	saddle	saddle	unstable	saddle	unstable	saddle
$(0, -\frac{\beta}{\rho\alpha})$			saddle	saddle	saddle	
(1,0)	saddle	saddle	unstable	saddle	unstable	saddle
(1,1)	stable	unstable	unstable	unstable	unstable	unstable
$(1, -\frac{(1-\rho)\alpha + \beta}{\rho\alpha})$			saddle		saddle	saddle
$(-\frac{\beta}{\rho\alpha}, 0)$			saddle	saddle	saddle	
$(-\frac{(1-\rho)\alpha + \beta}{\rho\alpha}, 1)$			saddle		saddle	saddle
$(-\frac{\beta}{\alpha}, -\frac{\beta}{\alpha})$			stable	stable	stable	stable

	$a > c$ and $b < d$			
	$\alpha + 2\beta < 0,$ $\rho > -\frac{\beta}{\alpha}$	$\alpha + 2\beta < 0,$ $\frac{\alpha + \beta}{\alpha} < \rho < -\frac{\beta}{\alpha}$	$\alpha + 2\beta > 0,$ $\rho > \frac{\alpha + \beta}{\alpha}$	$\alpha + 2\beta > 0,$ $-\frac{\beta}{\alpha} < \rho < \frac{\alpha + \beta}{\alpha}$
(0,0)	stable	stable	stable	stable
(0,1)	stable	saddle	stable	saddle
$(0, -\frac{\beta}{\rho\alpha})$	saddle		saddle	saddle
(1,0)	stable	saddle	stable	saddle
(1,1)	stable	stable	stable	stable
$(1, -\frac{(1-\rho)\alpha + \beta}{\rho\alpha})$	saddle	saddle	saddle	
$(-\frac{\beta}{\rho\alpha}, 0)$	saddle		saddle	saddle
$(-\frac{(1-\rho)\alpha + \beta}{\rho\alpha}, 1)$	saddle	saddle	saddle	
$(-\frac{\beta}{\alpha}, -\frac{\beta}{\alpha})$	unstable	unstable	unstable	unstable

the combined effect of interaction probabilities and the initial state of the population, different strategic types can peacefully inhabit the whole population, resulting in a diverse, yet harmonious society. Furthermore, for the latter two cases with invertible interaction probability matrix, we provide a sufficient and necessary condition for the system to attain a consensus of coexistent state in all communities. We hope that the present study might shed some light on understanding the effects of community structure, ubiquitous in the real world, on the evolutionary outcome.

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APPENDIX A: TABLE OF STABILITY OF EACH FIXED POINT IN DIFFERENT CASES

The stability of each fixed point in different cases is listed in Table I.

APPENDIX B: STABILITY OF EQUILIBRIA IN THE CASE OF $n=2$

Elements of the Jacobian matrix J of Eq. (2) are given by

$$J_{ii} = (1 - 2x_i) \left(\alpha \sum_{j=1}^n w_{ij} x_j + \beta \right) + \alpha w_{ii} x_i (1 - x_i),$$

$$J_{ij} = \alpha w_{ij} x_i (1 - x_i), \quad j \neq i. \tag{B1}$$

Under the condition of $n=2$, the Jacobian matrix J is reduced to

$$\begin{bmatrix} (1-2x_1)[\alpha(w_{11}x_1 + w_{12}x_2) + \beta] + \alpha w_{11}x_1(1-x_1) & \alpha\rho x_1(1-x_1) \\ \rho x_2(1-x_2) & (1-2x_2)[\alpha(w_{21}x_1 + w_{22}x_2) + \beta] + \alpha w_{22}x_2(1-x_2) \end{bmatrix}. \quad (\text{B2})$$

For each equilibrium, the corresponding eigenvalues of the Jacobian matrix J are listed in Table II.

(a) If $a > c$ and $b > d$, the equilibria $(0, -\beta/\rho\alpha)$, $\{1, -[(1-\rho)\alpha + \beta]/\rho\alpha\}$, $(-\beta/\rho\alpha, 0)$, $\{-[(1-\rho)\alpha + \beta]/\rho\alpha, 1\}$, and $(-\beta/\alpha, -\beta/\alpha)$ cannot be found in S_2 . There are only four boundary equilibria $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. At the equilibrium $(0, 0)$, the eigenvalues of the Jacobian matrix J are both $b-d > 0$, which results in an unstable equilibrium. Similarly, under the condition of $a > c$, the equilibrium $(1, 1)$ is a stable node. At the equilibria $(0, 1)$ and $(1, 0)$, the eigenvalues $\alpha(1-\rho) + \beta$ and $-\alpha\rho - \beta$ can be expressed by $(a-c)(1-\rho) + (b-d)\rho$ and $-(a-c)\rho + (\rho-1)(b-d)$, respectively. Obviously, the first one is positive and the second one is negative. In this case, both $(0, 1)$ and $(1, 0)$ are saddle nodes. Hence, the stability of the four equilibria implies that the population ends up with the all A state.

(b) If $a < c$ and $b < d$, the situation is similar to the above discussion.

(c) If $a < c$ and $b > d$, all the nine fixed points can be observed in the phase plane. As $-\beta/\alpha$ is always in the interval $(0, 1)$ and the two corresponding eigenvalues are both negative, the interior equilibrium $(-\beta/\alpha, -\beta/\alpha)$ exists and attracts every orbit in S_2 .

Note that $0 < -\beta/\rho\alpha < 1$ needs $\rho > -\beta/\alpha$, and $0 < -[(1-\rho)\alpha + \beta]/\rho\alpha < 1$ needs $\rho > (\alpha + \beta)/\alpha$. By comparing $-\beta/\alpha$ with $(\alpha + \beta)/\alpha$, we distinguish two situations as follows.

(1) Given $\alpha + 2\beta < 0$, we have $-\beta/\alpha < (\alpha + \beta)/\alpha$. Thus, there are three scenarios $\rho > (\alpha + \beta)/\alpha$, $-\beta/\alpha < \rho < (\alpha + \beta)/\alpha$ and $\rho < -\beta/\alpha$. In the first case, all the nine equilibria exist. We obtain that both of the eigenvalues corresponding to the points $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ are positive. Therefore, these four points are all unstable nodes. Furthermore, as one of the eigenvalues is negative and the other one is positive, the other fixed points except the interior one are all saddle nodes. In the second case, two fixed points $\{1, -[(1-\rho)\alpha + \beta]/\rho\alpha\}$ and $\{-[(1-\rho)\alpha + \beta]/\rho\alpha, 1\}$ are eliminated from the phase plane. By computing the two eigenval-

ues corresponding to each fixed point listed in Table II, we conclude that only the equilibria $(0, 0)$ and $(1, 1)$ are unstable nodes, the other equilibria except the interior fixed point are all saddle nodes. In the last case, as there are $-\beta/\alpha < 1/2$ and $\rho > 1/2$, the condition $\rho < -\beta/\alpha$ cannot be satisfied and this case is invalid.

(2) Given $\alpha + 2\beta > 0$, we have $-\beta/\alpha > (\alpha + \beta)/\alpha$. Thus, there are still three scenarios $\rho > -\beta/\alpha$, $(\alpha + \beta)/\alpha < \rho < -\beta/\alpha$ and $\rho < (\alpha + \beta)/\alpha$. In the first case, all the nine equilibria are in the phase plane and the population dynamics is similar to that under the conditions of $\alpha + 2\beta < 0$ and $\rho > (\alpha + \beta)/\alpha$. In the second case, the fixed points $(-\beta/\rho\alpha, 0)$ and $(0, -\beta/\rho\alpha)$ do not exist. Therefore, there are seven equilibria. Similarly, we find that the points $(0, 0)$ and $(1, 1)$ are unstable nodes and the other four are saddle nodes. In the third case, as the condition $\alpha + 2\beta > 0$ results in $(\alpha + \beta)/\alpha < 1/2$, inequality $\rho < (\alpha + \beta)$ contradicts the definition of the interaction probability ρ . Thus, this case does not exist.

(d) If $a > c$ and $b < d$, the analysis is similar to the discussion (c).

APPENDIX C: STABILITY OF THE INTERIOR EQUILIBRIUM IN n COMMUNITIES

Proof of theorem 1. Let $\Lambda(x_i)$ denote $1 - x_i$ and $G_i(\vec{x})$ denote $x_i(\alpha \sum_{j=1}^n w_{ij}x_j + \beta)$. Equations (2) and (3) are transformed to

$$\dot{x}_i = \Lambda(x_i)G_i(\vec{x}) \quad (\text{C1})$$

and

$$\dot{z}_i = G_i(\vec{z}). \quad (\text{C2})$$

By solving the system of linear equations

TABLE II. Eigenvalues of Jacobian matrix J .

equilibrium	eigenvalue λ_1	eigenvalue λ_2
$(0,0)$	$b-d$	$b-d$
$(0,1)$	$\alpha(1-\rho) + \beta$	$-\alpha\rho - \beta$
$(0, -\beta/\rho\alpha)$	$\beta(2\rho-1)/\rho$	$-\beta(1+\beta/\rho\alpha)$
$(1,0)$	$-\alpha\rho - \beta$	$\alpha(1-\rho) + \beta$
$(1,1)$	$c-a$	$c-a$
$\{1, -[(1-\rho)\alpha + \beta]/\rho\alpha\}$	$(a-c)(1-2\rho)/\rho$	$-[(1-\rho)\alpha + \beta](a-c)/\rho\alpha$
$(-\beta/\rho\alpha, 0)$	$-\beta(\rho\alpha + \beta)/\rho\alpha$	$\beta(2\rho-1)/\rho$
$\{-[(1-\rho)\alpha + \beta]/\rho\alpha, 1\}$	$-[(1-\rho)\alpha + \beta](a-c)/\rho\alpha$	$(a-c)(1-2\rho)/\rho$
$(-\beta/\alpha, -\beta/\alpha)$	$-\beta(a-c)/\alpha$	$-\beta(a-c)(2\rho-1)/\alpha$

$$\alpha \sum_{j=1}^n w_{ij} x_j + \beta = 0,$$

we obtain a solution $\vec{x}^* = (-\beta/\alpha, \dots, -\beta/\alpha)$ which is the interior equilibrium for both Eqs. (C1) and (C2).

Substituting \vec{x}^* into the Jacobian matrix of Eqs. (2) and (3), respectively, we obtain

$$\begin{aligned} (J_1)_{ij} &= \left. \frac{\partial \Lambda(x_i) G_i(\vec{x})}{\partial x_j} \right|_{\vec{x}^*} = \Lambda(x_i) \left. \frac{\partial G_i(\vec{x})}{\partial x_j} \right|_{\vec{x}^*} \\ &= \Lambda(x_i) \alpha w_{ij} x_i^* = -(1 + \beta/\alpha) \beta w_{ij}, \end{aligned} \quad (C3)$$

$$(J_2)_{ij} = \left. \frac{\partial G_i(\vec{z})}{\partial z_j} \right|_{\vec{x}^*} = \alpha w_{ij} x_i^* = -\beta w_{ij}. \quad (C4)$$

As $\alpha \sum_{j=1}^n w_{ij} x_j^* + \beta = 0$, we omit the term $\left. \frac{\partial \Lambda(x_i)}{\partial x_j} \right|_{\vec{x}^*} G_i(\vec{x}^*)$ in Eq. (C3) and the term $\alpha \sum_{j=1}^n w_{ij} x_j^* + \beta$ in Eq. (C4) when $i=j$. Obviously, there is $J_1 = (1 + \beta/\alpha) J_2$. Consequently, if λ is the eigenvalue of Jacobian matrix J_2 , $(1 + \beta/\alpha)\lambda$ is the eigenvalue of J_1 . As $1 + \beta/\alpha$ is positive, the stability of the equilibrium \vec{x}^* in Eqs. (2) and (3) are the same.

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